Remarks on The Entropy of 3-Manifolds

Bergfinnur Durhuus[1]

Matematisk Institut, University of Copenhagen
Universitetsparken 5, 2100 Copenhagen Ø
Denmark

Thordur Jonsson[2]

Raunvisindastofnun Haskolans, University of Iceland
Dunhaga 3, 107 Reykjavik
Iceland

Abstract. We give a simple combinatoric proof of an exponential upper bound on the number of distinct 3-manifolds that can be constructed by successively identifying nearest neighbour pairs of triangles in the boundary of a simplicial 3-ball and show that all closed simplicial manifolds that can be constructed in this manner are homeomorphic to S^3. We discuss the problem of proving that all 3-dimensional simplicial spheres can be obtained by this construction and give an example of a simplicial 3-ball whose boundary triangles can be identified pairwise such that no triangle is identified with any of its neighbours and the resulting 3-dimensional simplicial complex is a simply connected 3-manifold.

---

[1]e-mail: durhuus@math.ku.dk
[2]e-mail: thjons@raunvis.hi.is

1 Introduction

A few years ago a model based on random simplicial 3-manifolds was introduced as a discretization of 3-dimensional quantum gravity [1] in the same spirit as randomly triangulated surfaces have been used to study quantum gravity in 2 dimensions [2, 3, 4]. This model has been discussed by several authors, generalized to higher dimensions and extensively simulated, see e.g. [5, 6, 7, 8, 9, 10, 11, 12].

It has not yet been proven that these models have a convergent grand canonical partition function. The problem is to prove that the number $\mathcal{N}(N)$ of combinatorically distinct, but topologically identical, simplicial manifolds that can be constructed using N tetrahedra (or d-simplexes in d dimensions) satisfies an exponential bound

$$\mathcal{N}(N) \leq C^N \tag{1}$$

for some constant C. Numerical simulations indicate though that the bound holds, at least in 3 dimensions [13, 14]. In [1] some sufficient technical conditions for the exisitence of the desired bound for $d = 3$ were discussed.

Considering the general interest in and importance of this question, see [14, 15, 16, 17], we find it worthwhile to report on some partial results towards proving (1) and point out the principal obstacle to completing the argument. In particular, we show that a claimed proof [17] of (1) is based on a false assumption about the structure of simplicial 3-manifolds to which we provide a counterexample. For the 3-manifolds that satisfy this assumption we give a simple proof of an exponential bound for $\mathcal{N}(N)$.

2 An exponential bound

Any closed simplicial 3-manifold can be constructed by taking a suitable simplicial ball and identifying the triangles in the boundary pairwise. The resulting simplicial complex is a manifold if and only if its Euler characteristic is 0 [18]. The Euler characteristic of a 3-dimensional simplicial complex M is defined as

$$\chi(M) = \sum_{i=0}^{3}(-1)^i N_i(M) \tag{2}$$

where $N_i(M)$ is the number of i-simplexes in the complex.

It follows from this that one can construct any simplicial 3-manifold by first taking a tetrahedron, gluing on it another tetrahedron along a common triangle and so on until there are no more tetrahedra to be added. In this way one obtains a simplicial ball. Then one identifies pairwise the triangles in the boundary of the

resulting simplicial ball. It is not hard to prove, and we shall give the argument below, that the number of distinct simplicial balls that can arise in the first step of this construction increases at most exponentially with the number of tetrahedra. In general the number of triangles in the boundary of the simplicial ball is of the order of the number of tetrahedra and the possibility of superexponential factors arises when these triangles are identified pairwise. If one does not put any restriction on the topology of the resulting manifold the number of distinct manifolds does indeed grow factorially with the number of tetrahedra [1].

The Euler characteristic of a three-dimensional simplicial ball is 1. Suppose we are given such a ball B and a pairwise identification of its boundary triangles so that we obtain a simplicial complex M after the identifications have been carried out. After the identifications the triangles that were originally in the boundary of B form a two-dimensional complex K inside M. It is easily seen that the two-dimensional Euler characteristic, $\chi_2(K)$, of the complex K, defined as

$$\chi_2(K) = \sum_{i=0}^{2} (-1)^i N_i(K), \tag{3}$$

equals $1 + \chi(M)$ so M is a manifold if and only if $\chi_2(K) = 1$. Moreover, the first homotopy group of M is identical to the first homotopy group of the complex K [18].

Let us consider the problem of constructing a 3-manifold with the topology of the three-dimensional sphere S^3. Suppose we are given a simplicial ball and we want to identify the triangles in its surface pairwise so that we get a manifold homeomorphic to S^3. One of the conditions that the identification of triangles must satisfy (and the only condition if the Poincaré conjecture holds) is that noncontractible loops should not arise in the resulting complex. One would like to exhibit a local condition on the identifications that ensures this property and at the same time allows the construction of all possible simplicial spheres. For the analogous construction of S^2 by gluing 1-simplexes together this problem is solved (see [19]) by allowing only successive identifications of neighbouring links, i.e. pairs of links sharing a vertex. It is therefore reasonable to conjecture that for S^3 one should require that at each step identification of two triangles is only allowed if they share a link. More precisely, we say that a simplicial 3-manifold M has a *local construction* if there is a sequence of simplicial manifolds $T_1, \ldots, T_n$ such that

(i) T_1 is a tetrahedron

(ii) T_{i+1} is constructed from T_i by either gluing a new tetrahedron to T_i along one of the triangles in the boundary ∂T_i of T_i or by identifying a pair of nearest neighbour triangles in ∂T_i, i.e. two triangles sharing a link in ∂T_i.

(iii) $T_n = M$

We shall denote by $\mathcal{L}$ the collection of all closed simplicial 3-manifolds that have a local construction. In the next section we show that all manifolds in $\mathcal{L}$ are homeomorphic to S^3. It is, however, not known whether all simplicial spheres have a local construction. This was claimed for all simply connected 3-manifolds in [17] but as we shall see below the argument given there is not complete.

First we give a simple proof that the number of complexes in $\mathcal{L}$ with a given number N of 3-simplexes is bounded by an exponential function of N.

Theorem 1 *There is a constant C such that*

$$\#\{M \in \mathcal{L} : N_3(M) = N\} \le C^N. \tag{4}$$

Proof. It is easy to see that if a manifold has a local construction and contains N tetrahedra it also has a local construction where all the tetrahedra are assembled in the first N steps. In the first N steps we therefore build a tree-like 3-manifold with at most 4 branches emanating from each node. In the dual picture (where a tetrahedron is a vertex and a pair of identified triangles is a link joining the corresponding vertices) such a 3-manifold is a φ^4 tree-graph with N vertices. The number of distinct graphs of that kind is well-known to be bounded by C_1^N where C_1 is a constant.

Let us now assume that we have one of the tree-like simplicial manifolds described above made of N tetrahedra. Its surface consists of $2N + 2$ triangles. We wish to estimate from above how many ways there are to close up this manifold by successively identifying nearest neighbour pairs of triangles. Suppose that in the beginning there are n_1 pairs of nearest neighbour triangles that are to be identified. Once these identifications have been carried out the remaining unidentified triangles in general have new neighbours and there are n_2 pairs of nearest neighbour triangles that are to be identified. We continue in this fashion, identifying n_i pairs in the i-th step until there are no triangles left after f steps. Clearly $f \le N+1$ because $n_i \ge 1$.

The number of ways to choose the n_1 pairs of triangles that participate in the first round of identifiactions is bounded by

$$\binom{2N + 2}{n_1} 3^{n_1}. \tag{5}$$

After carrying out these identifications there arise at most $2n_1$ new pairs of nearest neighbour triangles that might be identified, see Fig. 1.

In the next step of the construction we choose n_2 triangles out of the $4n_1$ triangles that possibly may be identified with one of their neighbours and identify each of

4

them with one of their neighbours. The number of ways this can be done is bounded by

$$\binom{4n_1}{n_2} 3^{n_2}. \tag{6}$$

We continue in this fashion until there are no triangles left. Clearly

$$2 \sum_{i=1}^{f} n_i = 2N + 2 \tag{7}$$

where $2n_f$ is the number of triangles left before the final step in the identification process is carried out.

The total number of ways one can close the tree-like manifold is therefore bounded by

$$
\begin{aligned}
\sum_{f=1}^{N+1} \sum_{n_1,\dots,n_f} & \binom{2N+2}{n_1}\binom{4n_1}{n_2}\binom{4n_2}{n_3}\dots\binom{4n_{f-1}}{n_f} 3^{N+1} \\
\leq\ & \sum_{f=1}^{N+1} \binom{N}{f-1} 2^{6N+6} 3^{N+1} \\
\leq\ & \sum_{f=1}^{N+1} 2^{7N+6} 3^{N+1} \\
\leq\ & C_2^N.
\end{aligned}
\tag{8}
$$

This completes the proof with $C = C_1 C_2$.

Note that in the bound derived above we have not used that $\chi = 0$ for 3-manifolds so we have in fact established a bound on the number of pseudomanifolds with a local construction. A bound of the form (4) is also obtained in [17] but with a more elaborate proof.

3 Properties of manifolds with a local construction

We begin by demonstrating that the elements in $\mathcal{L}$ are simplicial spheres. This is a consequence of the following result.

Theorem 2 *Let $T_1, \dots, T_n$ be a local construction of a simplicial manifold M. Then, for all $i = 1, \dots, n$, T_i is homeomorphic to S^3 with a number of simplicial 3-balls removed. The boundary ∂T_i is a union of simplicial 2-spheres, $S_1, \dots, S_k$, and these fulfill:*

(i) S_r and S_s have at most one point (vertex) in common for $r \neq s$, $1 \leq r, s \leq k$

(ii) The connected components of ∂T_i are simply connected.

Proof. This is a straightforward inductive argument. The single tetrahedron T_1 is homeomorphic to a closed 3-ball which may be regarded as S^3 with an open 3-ball removed.

Assume that T_i satisfies the properties listed in the theorem. If T_{i+1} is obtained from T_i by gluing on a tetrahedron this clearly does not change the homeomorrphism class of the manifold and T_{i+1} has the properties listed in the theorem. On the other hand, when gluing together two neighbouring triangles in ∂T_i to obtain T_{i+1} a number of distinct possibilities have to be considered. Note, however, that in all cases the two triangles belong to the same 2-sphere $S_r \subset \partial T_i$ due to condition (i).

(a) The two triangles have only one link in common. Thus, the two different vertices P_1 and P_2 in ∂T_i opposite to this link are identified. In this case the sphere S_r shrinks to a new sphere containing 2 fewer triangles. Those of the other spheres that have P_1 or P_2 as a contact point with S_r now all have a common contact point with the new sphere. Because of (ii) this is the only point shared by any two of the spheres involved and clearly the boundary remains simply connected.

(b) The two triangles have one link and the point P opposite to the link in common. In this case the sphere S_r splits into two spheres with one point in common, namely P. It is easy to check that conditions (i) and (ii) still hold.

(c) The two triangles have two links in common and these two links emerge from a vertex P. In this case the boundary spheres touching S_r at P are split off when we identify the triangles and S_r shrinks to a new sphere. This gives rise to at most one more connected component in the boundary of the manifold and conditions (i) and (ii) are still satisfied.

(d) The two triangles have all three edges in common. In this case S_r consists solely of these two triangles and S_r disappears upon their identification. The boundary will in general split into three parts but still satisfies (i) and (ii).

As mentioned above we do not have a proof that every simplicial 3-sphere has a local construction. Given a simplicial 3-sphere M made of N tetrahedra a possible strategy to produce such a local construction is first to assemble all the N tetrahedra as described in the proof of Theorem 1 in the previous section and then successively identify nearest neighbour triangles. Continuing in this fashion one ends up with a simplicial manifold M' which has the properties described in Theorem 2 and no two neighbouring triangles in $\partial M'$ may be identified in order to construct M. One must then show that the requirement that M be homemorphic to S^3 implies that $M' = M$, i.e. $\partial M' = \emptyset$. The following result is a step in this direction.

Proposition 3 *Let M' be any simplicial 3-manifold with a nonempty boundary hav-*

ing the properties described in Theorem 2. Let M be a closed 3-manifold constructed from M' by identifying the triangles in $\partial M'$ pairwise in such a way that at least two triangles from different boundary spheres S_r and S_s are identified. Then M is not homeomorphic to S^3.

Proof. Let S_r be a 2-sphere in the boundary of M' and choose a collar neighbourhood C_r of S_r which may be pinched at the finite number of points where S_r meets other boundary spheres. The boundary of C_r consists of S_r and another 2-sphere $\bar{S}_r$ which lies in the interior of M' except at the pinching points which it shares with S_r. Suppose now that a triangle t in S_r is identified with another triangle t' in S_s, $r \neq s$. Consider a smooth curve connecting a point p in t with the corresponding point p' in t'. We can assume that the curve intersects $\bar{S}_r$ exactly once. In M this curve will be closed since the endpoints have been identified, and it intersects $\bar{S}_r$ once. If M is homeomorphic to S^3 this contradicts the Jordan-Brouwer theorem [20], which states that any 2-sphere in S^3 separates it into two connected components and hence any smooth closed curve which intersects the 2-sphere transversally must intersect it an even number of times. The fact that in the present case the sphere may be pinched at a finite number of points does not affect the argument.

We remark that the proof above uses explicitly that that M is homeomorphic to S^3 and not only that M is simply connected. Proposition 3 implies that in $\partial M'$ only identifications of triangles within the same 2-sphere in the boundary are allowed if the resulting closed manifold M is homeomorphic to S^3. In order to establish the existence of a local construction for all simplicial 3-spheres it would therefore be sufficient to prove the following: If M is obtained from M' by identifications of boundary triangles in such a way that any pair of identified triangles sits within the same 2-sphere in $\partial M'$ and there is some 2-sphere $\partial M'$ in which no two neighbouring triangles are identified, then M is not homeomorphic to S^3. We shall in the next section show that this does not hold by exhibiting an example of a simplicial manifold M' homeomorphic to a 3-ball and a pairwise identification of the triangles in its boundary such that no triangle is identified to a neighbour but nevertheless the resulting manifold is simply connected and consequently homeomorphic to S^3 provided the Poincaré conjecture holds. This does not prove the impossibility of a local construction for this particular manifold, but certainly implies that the ordering of identifications in the gluing process must be chosen with care if one is to have a local construction.

4   A counterexample

Let us now explain the example mentioned above. It is a little complicated so we shall explain it in a few steps with the aid of diagrams. First take two tetrahedra and glue them together along one triangle. The surface of the resulting ball is a triangulation of S^2 consisting of 6 triangles. Now cut this triangulation open along one of the boundary links which joins a vertex of degree 3 with a vertex of degree 4. Glue to this surface a pair of triangles that are glued to each other along two links. Now we have a triangulation T_1 of S^2 consisting of 8 triangles.

Next cut the triangulation T_1 open along the links a, b and c as indicated in Fig. 2. In this way we obtain a two-dimensional triangulation T_2 with the topology of a disc and a boundary consisting of 6 links l_1, l_2, l_3, l_4, l_5 and l_6. We let l_1 and l_2 be the two links that arise when we cut along a, l_3 and l_4 correspond in the same way to b and l_5 and l_6 correspond to c. In Fig. 3 we have drawn the dual diagram of this triangulation where a triangle is denoted by a dot and two dots are connected by a link if and only if the corresponding triangles share a link. The boundary links correspond to external legs on the dual diagram. We label the triangles A, B, C, D, E, F, G and H as indicated in Fig. 3.

Next take another pair of tetrahedra and carry out exactly the same operations, obtaining another triangulation T_2' of the disc with boundary links $l_1', \ldots, l_6'$ where l_i' has the same position on the new triangulation as l_i on the first one and in the same fashion we label the triangles A', B' etc.

Now glue the triangulations T_2 and T_2' together along their boundaries so that l_i' is identified with l_i, $i = 1, \ldots, 6$. Then we obtain a triangulation T_3 of S^2 consisting of 16 triangles, see Fig. 4. This triangulation can be extended to a triangulation of the three-dimensional ball, e.g. by placing one vertex in the interior of the ball and connecting this vertex to all the boundary vertices by links. Now identify the triangles in the boundary in the following way:

$$A = C, \quad B = D, \quad F = G, \quad H = E \tag{9}$$

and correspondingly for the primed triangles in T_3. Of course two triangles can be identified in 3 ways, but the identification is uniquely determined by specifying two boundary links, one in each triangle, that are identified. For A and C we identify l_1 with l_2, for B and D we identify l_3 with l_4 and for F and G we identify l_5 with l_6. After these identifications have been carried out there is only one way to identify H with E because their boundary links have already been identified. The same method is used in the identification of the primed triangles.

It is clear from Fig. 4 that no two triangles that share a link have been identified.

It is a straightforward counting problem to calculate the Euler characteristic of the complex K that arises from the identifications (9) and their primed counterpart. We find

$$\chi(K) = 1, \tag{10}$$

and hence the 3-dimensional simplicial complex we have constructed is a manifold.

We have drawn a picture of one half of the complex K in Fig. 5, i.e. the half that is made up of the unprimed triangles. Let us call this half $\bar{K}$. In order to see that K is simply connected one can argue as follows: Let δ be a curve in $\bar{K}$ such that the endpoints of δ lie in the boundary of $\bar{K}$, where by the boundary of $\bar{K}$ we mean the links where $\bar{K}$ meets the other half of the complex K, see Fig. 5. It is easy to see that there is a homotopy in $\bar{K}$, which keeps endpoints fixed, from δ to another curve δ' which lies entirely in the boundary of $\bar{K}$. It follows that any closed curve in K is homotopic to a closed curve in the boundary of $\bar{K}$. The boundary of $\bar{K}$ is contractible so K is simply connected and therefore the manifold is also simply connected. This completes the discussion of the counterexample.

We remark that in the above example we allowed two triangles to be glued along two links. The example can easily be generalized so that no pair of triangles shares more than one link, by subdividing the triangles G and H and their primed counterparts.

5 Discussion

In this note we have discussed a possible strategy to prove an exponential bound on the number of combinatorially distinct simplicial 3-spheres as a function of volume (= number of 3-simplexes). We believe that this method may be useful for future investigations. Partial results towards a proof have been obtained and the principal obstacle to completing the proof has been described.

We would like to mention that a proof of the local constructibility of all simplicial 3-spheres will, as a consequence of Theorem 2, yield an an algorithm for constructing all simplicial 3-spheres with a given number of tetrahedra and hence imply that S^3 is algorithmically recognizable. The algebraic recognizability of S^3 was proven recently [21] and can perhaps be viewed as an indication that the strategy outlined here can be implemented. In 5 and more dimensions it is known that spheres are not algorithmically recognizable [22] and there may be problems in 4 dimensions as well [23]. It is therefore likely that a different method will be needed to establish an exponential bound on $\mathcal{N}(N)$ in 4 and more dimensions if it holds in these dimensions at all.

Finally, it should be noted that proving the local constructibility of all simply connected simplicial 3-manifolds is a far more ambitious project than proving this for manifolds with the topology of S^3. By Theorem 2 such a result would imply the Poincaré conjecture.

References

[1] J. Ambjørn, B. Durhuus, T. Jonsson, Mod. Phys. Lett. A6 (1991) 1133.

[2] J. Ambjørn, B. Durhuus and J. Fröhlich, Nucl. Phys. B 257 (1985) 433.

[3] F. David, Nucl. Phys. B 257 (1985) 45, 543.

[4] V. A. Kazakov, I. K. Kostov and A. A. Migdal, Phys. Lett. 157B (1985) 295.

[5] M. E. Agishtein and A. A. Migdal, Mod. Phys. Lett. A6 (1991) 1863.

[6] J. Ambjørn and S. Varsted, Phys. Lett. 226B (1991) 285.

[7] S. Catterall, J. Kogut and R. Renken, Phys. Lett. B328 (1994) 277.

[8] M E. Agishtein and A. A. Migdal, Nucl. Phys. B 385 (1992) 395.

[9] N. Godfrey and M. Gross, Phys. Rev. D 43 (1991) 1749.

[10] B. Brugman and E. Marinari, Phys. Rev. Lett. 70 (1993) 1908.

[11] D. V. Boulatov and A. Krzywicki, Mod. Phys. Lett. A 6 (1991) 3005.

[12] J. Ambjørn, D. V. Boulatov, A. Krzywicki and S. Varsted, Phys. Lett. B 276 (1992) 432.

[13] J. Ambjørn and S. Varsted, Nucl. Phys. B 373 (1992) 557.

[14] S. Catterall, J. Kogut and R. Renken, *Entropy and the approach to the thermodynamic limit in three-dimensional simplicial gravity*, CERN-TH 7404/94.

[15] S. Catterall, J. Kogut and R. Renken, *On the absence of exponential bound in four dimensional simplicial gravity*, CERN-TH-7197/94.

[16] J. Ambjørn and J. Jurkiewicz, *On the exponential bound in four dimensional simplicial gravity*, NBI-HE-94-29.

[17] D. V. Boulatov, *On entropy of 3-dimensional simplicial complexes*, NBI-HE-94-37.

[18] H. Seifert and W. Threlfall, *A textbook of topology.* Academic Press, Orlando (1980).

[19] B. Durhuus, In: *Constructive quantum field theory II*, G. Velo and A. Wightman eds., Plenum Press, 1990.

[20] M Spivak, *Differential geometry*, Publish or Perish, Boston, 1972.

[21] J. H. Rubinstein, *An algorithm to recognise the 3-sphere*, Lecture at ICM, Zürich, 1994. A. Thompson, *Thin position and the recognition problem for S^3*, preprint.

[22] I. A. Volodin, V. E. Kuznetsov and A. T. Fomenko, Russian Math. Surveys, 29:5 (1974) 71.

[23] A. Nabutovsky and R. Ben-Av, Commun. Math. Phys. 157 (1993) 93.

Figure caption.

Fig. 1. After identifying the triangles A and A' the triangles B and D become nearest neighbours and the same applies to the triangles C and E. They may therefore be identified once A and A' have been glued together.

Fig. 2. The triangulation T_1. We cut it open along the links marked a, b and c.

Fig. 3 The dual graph of the triangulation T_2. Vertices in the dual graph correspond to triangles in the triangulation and external legs correspond to boundary links.

Fig. 4. The dual graph corresponding to the triangulation T_3.

Fig. 5. One half of the complex K. This subcomplex is made up of the 4 triangles that correspond to the original 8 unprimed triangles after they have been identified pairwise. There is another identical subcomplex corresponding to the primed triangles which is glued to this half along the links α, β and γ. The shaded triangle in the Figure lies on top of another triangle and is glued to the triangles below along two of its boundary links.

This figure "fig1-1.png" is available in "png" format from:

http://arXiv.org/ps/hep-th/9410110v1

This figure "fig2-1.png" is available in "png" format from:

http://arXiv.org/ps/hep-th/9410110v1

This figure "fig1-2.png" is available in "png" format from:

http://arXiv.org/ps/hep-th/9410110v1

This figure "fig2-2.png" is available in "png" format from:

http://arXiv.org/ps/hep-th/9410110v1

This figure "fig2-3.png" is available in "png" format from:

http://arXiv.org/ps/hep-th/9410110v1